Resonant Modes in Brane-World Inflation

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Contents

Motivation

- There has been considerable interest in brane-world models recently, where ordinary matter is confined to a subspace of a higher-dimensional spacetime, I will focus on the Randall–Sundrum model with a single brane.
- Minkowski and cosmological background solutions are known but cosmological perturbations are difficult.
- Understanding perturbations at high energies (inflation) is particularly important because the effects are potentially strong and testable in CMBR measurements.
- In this regard it is important to determine the quantum vacuum state for brane inflation, which has not yet been calculated. (David Wands's talk on Saturday)

Outline

- We sought first to investigate these using a toy model with coupled scalar fields in place of gravity and matter. [hep-th/0504201]
- These coupled boundary-bulk systems have a rich structure of behaviour.
- We then investigated cosmological scalar perturbations in the Randall–Sundrum model, focussing on sub-horizon perturbations for an inflating brane.
- We find modes associated with brane-world inflation that obey a modified dispersion relation.

Toy Model with Scalar Field

We consider a model with a fixed background and a bulk scalar field coupled to a brane scalar field. The action is:

$$\begin{aligned} S &= \frac{1}{2}\int_{\mathcal{V}} d^{d+1}x\sqrt{-G}\left(G^{MN}\phi_{,M}\phi_{,N} + m^2\phi^2\right) \\ &\quad -\frac{1}{2}\int_{\partial\mathcal{V}} d^d x\sqrt{-g}\left[g^{\mu\nu}q_{,\mu}q_{,\nu} + \mu^2 q^2 + 2\beta\phi q\right] . \end{aligned}$$

G_{MN} is the bulk metric.

$g_{\mu\nu}$ is the brane metric.

ϕ is the bulk scalar field with mass m.

q is the brane scalar field with mass μ.

β is the coupling between the fields.

Equations of Motion

The coupled wave equations are then

$$^{(d)}\Box q = \mu^2 q + \beta\phi|_{\rm brane} \,,$$

on the brane, and a free wave equation for ϕ in the bulk

$$^{(d+1)}\Box\phi = m^2\phi \,,$$

subject to the boundary condition

$$\phi'|_{\rm brane} = \frac{\beta}{2} q \,.$$

Solutions for a Minkowski Bulk

[A. George — hep-th/0412067]

First consider a flat brane located at $y = 0$ in a flat bulk

$$ds^2 = -dt^2 + \delta^{ij} dx_i dx_j + dy^2 \,,$$

The equations of motion are

$$\begin{aligned} \ddot{q} + k^2 q &= -\mu^2 q - \beta\phi|_{\rm brane} \,, \\ \ddot{\phi} + k^2\phi &= \phi'' - m^2\phi \,, \end{aligned}$$

Solve the bulk equation by separating variables

$$\ddot{T}_\rho + \omega^2 T = 0 \qquad \text{where} \qquad \omega^2 = \rho^2 + m^2 + k^2 \,,$$

$$q(t) = \int_0^\infty d\rho \, C_\rho T_\rho(t) \,,$$

$$\phi(t,z) = \int_0^\infty d\rho \, [A_\rho \cos(\rho y) + B_\rho \sin(\rho y)] \, T_\rho(t) \,.$$

The boundary field equation and junction condition relate the coefficients

$$2\rho(\rho^2 + m^2 - \mu^2)B_\rho = \beta(\rho^2 + m^2 - \mu^2)C_\rho = \beta^2 A_\rho\,.$$

Bound States:

We impose the condition $\Im(\rho) > 0$ and

$$A_\rho = -iB_\rho\,.$$

So that $\phi(y) \to e^{i\rho y}$, which is exponentially decaying. Combined with the condition above, this yields

$$2\rho(\rho^2 + m^2 - \mu^2) + i\beta^2 = 0\,.$$

This will describe stable oscillations if $\omega^2 > 0$, which requires

$$\beta^2 < 2m\mu^2\,.$$

- For massive oscillators we get a single bound state provided that the masses and coupling obey $\beta^2 < 2m\mu^2$.
- Compare to non-brane-world case of two coupled oscillators where we can diagonalize the mass matrix.
- The stable bound state allows us to define a vacuum state.

Solutions for an AdS Bulk

[Koyama, Mennim & Wands — hep-th/0504201]

The Randall–Sundrum model: a flat brane ($z = \ell$) in an AdS bulk

$$ds^2 = \frac{\ell^2}{z^2}\left(-dt^2 + \delta^{ij} dx_i dx_j + dz^2\right) ,$$

The equations of motion are

$$\begin{aligned} \ddot{q} + k^2 q &= -\mu^2 q - \beta\phi|_{\mathrm{brane}} , \\ \ddot{\phi} + k^2\phi &= \phi'' - \frac{3}{z}\phi' - \frac{m^2\ell^2}{z^2}\phi , \end{aligned}$$

Separating variables allows us to write the solution as

$$\begin{aligned} q(t) &= \int_0^\infty d\rho\, C_\rho \mathrm{e}^{\pm i\omega t} , \\ \phi(t,z) &= \int_0^\infty d\rho\, z^2 \left[A_\rho J_\nu(\rho z) + B_\rho Y_\nu(\rho z)\right] \mathrm{e}^{\pm i\omega t} . \end{aligned}$$

where $\omega^2 = \rho^2 + k^2$ and $\nu = \sqrt{4 + m^2\ell^2}$.

Bound State Solutions

Consider the simple case where $m = \mu = 0$, and use units where $\ell = 1$. Bound states satisfy

$$\rho^3 H_1^{(1)}(\rho) - \frac{\beta^2}{2} H_2^{(1)}(\rho) = 0,$$

- When $\beta = 0$ there is a zero-mode solution $\rho = 0$.
- When $\beta \neq 0$ this is no longer a solution; for small β, it becomes 4 roots, two of which satisfy $\Im\rho > 0$:

$$\begin{aligned} \rho_1 &\approx -\sqrt{\beta} + i\frac{\pi}{16}\beta^{3/2}, \\ \rho_2 &\approx i\sqrt{\beta} + \text{no real part}, \end{aligned}$$

Recall that $\omega = \pm\sqrt{\rho^2 + k^2}$.

- The imaginary root $\rho \approx i\sqrt{\beta}$ is a bound state, but leads to tachyonic instability on long wavelengths.
- The other root $\rho \approx -\sqrt{\beta} + i\pi\beta^{3/2}/16$ is a quasi-normal mode describing a metastable state.
- For massive fields we can calculate a critical coupling, as before, below which there is a bound state.

de Sitter Brane-World

The background line element has the form

$$\mathrm{d}s^2 = N(z)^2 \left[-\mathrm{d}t^2 + \mathrm{d}z^2 + e^{2Ht}\mathrm{d}\vec{x}^2 \right],$$

where H is the Hubble parameter (constant in a de Sitter universe), ℓ is the AdS curvature length-scale,

$$N(z) = \frac{H\ell}{\sinh(Hz)},$$

and the brane is located at

$$z_\mathrm{b} = H^{-1} \operatorname{arcsinh}(H\ell).$$

Perturbations

Cosmological perturbations are most easily expressed in terms of Mukohyama's master variable, Ω_{M}.

$$ds^2 = N(z)^2 \Big[-(1+2A)dt^2 + (1+2A_{zz})dz^2 + 2A_z dtdz + a^2(1+2R)\mathrm{d}\vec{x}^2 \Big] .$$

with

$$A = -\frac{1}{6aN^3}\left(2\Omega_{\mathrm{M}}'' - 3\frac{N'}{N}\Omega_{\mathrm{M}} + \ddot{\Omega}_{\mathrm{M}} - \frac{N^2}{\ell^2}\Omega_{\mathrm{M}}\right) ,$$

etc.

We perform a rescaling,

$$\Omega_{\mathrm{M}} = a^3 N^3 \Omega \,,$$

which obeys a canonical wave equation with a negative effective square-mass.

$$\frac{\partial^2\Omega}{\partial t^2} + 3H\frac{\partial\Omega}{\partial t} + k^2 e^{-2Ht}\,\Omega = \frac{\partial^2\Omega}{\partial z^2} - 3H\frac{\cosh(Hz)}{\sinh(Hz)}\,\frac{\partial\Omega}{\partial z} + \frac{4H^2}{\sinh^2(Hz)}\,\Omega \,.$$

We can solve this by separating variables, writing

$$\Omega(t,z) = \int \mathrm{d}\nu \; T_\nu(t) Z_\nu(z) \,.$$

$$\ddot{T} + 3H\dot{T} + \left[\left(\frac{9}{4} - \nu^2\right) H^2 + k^2 e^{-2Ht}\right] T = 0 \,,$$

$$Z'' - 3H\frac{\cosh(Hz)}{\sinh(Hz)} Z' + \left[\left(\frac{9}{4} - \nu^2\right) H^2 + \frac{4H^2}{\sinh^2(Hz)}\right] Z = 0 \,,$$

the solutions to which are

$$T_\nu(t) = e^{-3Ht/2} \left\{ A_\nu \, J_\nu\left(\frac{k}{H} e^{-Ht}\right) + B_\nu \, Y_\nu\left(\frac{k}{H} e^{-Ht}\right) \right\} ,$$

$$Z_\nu(z) = \sinh^2(Hz) \left\{ a_\nu \, P_{\nu - 1/2}\Big(\cosh(Hz)\Big) + b_\nu \, Q_{\nu - 1/2}\Big(\cosh(Hz)\Big) \right\} ,$$

where J and Y are Bessel functions and P and Q are associated Legendre functions.

Perturbations on the boundary

The boundary field is the perturbation of the inflaton ϕ, written as $\delta\phi$. However, it is more convenient to define the, gauge-invariant, Muhkanov-Sasaki variable Q, given by

$$Q = \delta\phi - \frac{\dot{\phi}}{H}\Psi$$

where Ψ is the Newtonian potential. The equation of motion for Q is

$$\ddot{Q} + 3H\dot{Q} + \left[k^2 e^{-2Ht} + (\eta - 6\varepsilon)H^2\right] Q = - \frac{\dot{\phi}k^2}{6H}\left\{\ddot{\Omega} + 5H\dot{\Omega} + \left[k^2 e^{-2Ht} + (6 - 3\varepsilon)H^2\right]\Omega\right\}$$

where

$$\varepsilon = -\frac{\dot{H}}{H^2} \qquad \text{and} \qquad \eta = \frac{V''}{H^2}$$

We seek a solution by writing Q in terms of the functions T_ν as

$$Q = \int \mathrm{d}\nu \, C_\nu T_\nu(t) \,,$$

Substituting this and the expression for Ω into the boundary equation of motion gives

$$\int \mathrm{d}\nu \left[\nu^2 - \frac{9}{4} - 6\varepsilon + \eta\right] H^2 C_\nu T_\nu =$$
$$- \frac{\beta k^2}{\kappa\sqrt{6H}} \int \mathrm{d}\nu \left\{2H\dot{T}_\nu + \left[\nu^2 - \frac{15}{4} - 3\varepsilon\right] H^2 T_\nu\right\} Z_\nu\Big|_{\mathrm{brane}}$$

where

$$\beta^2 = \frac{\kappa^2 \dot{\phi}^2}{6H} \approx \frac{\varepsilon}{3}$$

Large k limit

Changing to conformal time,

$$\int \mathrm{d}\nu \left[\nu^2 - \frac{9}{4} - 6\varepsilon + \eta\right] H^2 C_\nu T_\nu =$$

$$\frac{\beta k^2}{\kappa\sqrt{6H}} \int \mathrm{d}\nu \left\{ \frac{2H}{a}\frac{\mathrm{d}T_\nu}{\mathrm{d}\tau} - \left[\nu^2 - \frac{15}{4} - 3\varepsilon\right] H^2 T_\nu \right\} Z_\nu \Big|_{\mathrm{brane}}$$

In the small-wavelength limit we can treat a as approximately constant. Then we can write T_ν as

$$T_\nu(\tau) = e^{i\omega_\nu \tau}$$

where ω_ν must satisfy

$$\omega_\nu^2 - 2Hai\omega_\nu + \left(\nu^2 - \frac{9}{4}\right) H^2 a^2 - k^2 = 0$$

We are interested in the primordial universe where $H\ell \ll 1$, so we use the asymptotic expansion

$$\frac{Q_{\nu+1/2}\left(\sqrt{1+H^2\ell^2}\right)}{Q_{\nu-1/2}\left(\sqrt{1+H^2\ell^2}\right)} \sim \frac{2\nu+1}{4(\nu+1)}\frac{1}{H\ell}$$

This gives the condition as

$$\frac{\beta^2 k^2}{a^2 H^2}\left(\frac{2i\omega}{aH}-\nu^2\right) \approx -\left(\frac{i\omega}{aH}+3\right)\left(\nu^2-\frac{9}{4}\right)\left(\nu+\frac{1}{2}\right)$$

Defining $\bar{k} = k/aH$ and $\bar{\omega} = \omega/aH$, we have the following coupled system of equations

$$\bar{\omega}^2 - 2i\bar{\omega} + \nu^2 - \frac{9}{4} - \bar{k}^2 = 0\,,$$

$$2i\beta^2\bar{k}^2\bar{\omega} - \beta^2\bar{k}^2\nu^2 = -(i\bar{\omega}+3)\nu^3 - (i\bar{\omega}+3)\frac{\nu^2}{2}\,.$$

We can find the roots approximately by expanding in orders of k. Four of the roots are

$$\nu = \pm(1+i)\sqrt{\bar{k}} \pm \frac{i}{\beta^2\sqrt{\bar{k}}} - \frac{\bar{\omega}^2}{\beta^2\bar{k}^2}\,,$$

$$\bar{\omega} = \bar{k} + \frac{1+i}{\beta^2\sqrt{\bar{k}}} \qquad \text{or} \qquad -\bar{k} + \frac{1-i}{\beta^2\sqrt{\bar{k}}}\,,$$

Of these, two are normalizable.

$$\nu \approx (1+i)\sqrt{\bar{k}} + \frac{i}{\beta^2\sqrt{\bar{k}}} - \frac{\bar{\omega}^2}{\beta^2\bar{k}^2}\,,$$

$$\bar{\omega} \approx \bar{k} + \frac{1+i}{\beta^2\sqrt{k}} \qquad \text{or} \qquad -\bar{k} + \frac{1-i}{\beta^2\sqrt{k}}\,,$$

Two more roots are:

$$\nu \approx \mp i\beta^2\bar{k} + \frac{2}{\beta^2} - \frac{i}{2}$$

$$\bar{\omega} \approx \pm\bar{k}\sqrt{1+\beta^4}$$

These are both normalisable.

And a final two:

$$\nu \approx \mp i\bar{k} + \frac{3}{\beta^2} - \frac{i}{2}$$

$$\bar{\omega} \approx \pm i\beta^2\bar{k} - \frac{\beta^2}{2}$$

These are normalizable, but are rapidly growing and decaying.

We interpret the following modes as being of physical interest:

$$\nu \approx \pm(1+i)\sqrt{\bar{k}} \pm \frac{i}{\beta^2\sqrt{\bar{k}}} ,$$
$$\bar{\omega} \approx \pm\bar{k}$$

corresponding to a free bulk mode uncoupled to a perturbation of the Mukhanov–Sasaki variable Q.

$$\nu \approx \mp i\beta^2\bar{k} + \frac{2}{\beta^2} - \frac{i}{2}$$
$$\bar{\omega} \approx \pm\bar{k}\sqrt{1+\beta^4}$$

which is a perturbation of the brane and bulk fields.
Have a modified dispersion relation, although $\mathcal{O}(\varepsilon^2)$.
Compared with Transplankian modifications these are at a lower energy scale, and do not have to be added by hand.

Conclusions

- Coupled boundary bulk systems have a rich structure of behaviour
- Our toy models show bound states when the fields are massive and the coupling is less than a critical value.
- For scalar fields in a Randall–Sundrum background we can find quasi-normal modes.
- For metric perturbations in brane-world inflation we have determined bound states in the small-wavelength limit.
- There is a possibility of getting modified dispersion relations.
- Future work is to construct the quantum vacuum state for inflation and look for QNMs.