Resonant Modes in Brane-World Inflation

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Introduction

A toy model with a scalar field

de Sitter Brane-World

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Motivation

- There has been considerable interest in brane-world models recently, where ordinary matter is confined to a subspace of a higher-dimensional spacetime, I will focus on the Randall–Sundrum model with a single brane.
- Minkowski and cosmological background solutions are known but cosmological perturbations are difficult.
- Understanding perturbations at high energies (inflation) is particularly important because the effects are potentially strong and testable in CMBR measurements.
- In this regard it is important to determine the quantum vacuum state for brane inflation, which has not yet been calculated. (David Wands’s talk on Saturday)
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Outline

• We sought first to investigate these using a toy model with coupled scalar fields in place of gravity and matter. [hep-th/0504201]
• These coupled boundary-bulk systems have a rich structure of behaviour.
• We then investigated cosmological scalar perturbations in the Randall–Sundrum model, focussing on sub-horizon perturbations for an inflating brane.
• We find modes associated with brane-world inflation that obey a modified dispersion relation.
Toy Model with Scalar Field

We consider a model with a fixed background and a bulk scalar field coupled to a brane scalar field. The action is:

\[
S = \frac{1}{2} \int_V d^{d+1}x \sqrt{-G} \left( G^{MN} \phi_{,M} \phi_{,N} + m^2 \phi^2 \right) \\
- \frac{1}{2} \int_{\partial V} d^d x \sqrt{-g} \left[ g^{\mu \nu} q_{,\mu} q_{,\nu} + \mu^2 q^2 + 2\beta \phi q \right].
\]

\( G_{MN} \) is the bulk metric.
\( g_{\mu \nu} \) is the brane metric.
\( \phi \) is the bulk scalar field with mass \( m \).
\( q \) is the brane scalar field with mass \( \mu \).
\( \beta \) is the coupling between the fields.
Equations of Motion

The coupled wave equations are then

\[(d) \Box q = \mu^2 q + \beta \phi \big|_{\text{brane}},\]

on the brane, and a free wave equation for \(\phi\) in the bulk

\[(d+1) \Box \phi = m^2 \phi,\]

subject to the boundary condition

\[\phi' \big|_{\text{brane}} = \frac{\beta}{2} q. \]
Solutions for a Minkowski Bulk

[A. George — hep-th/0412067]

First consider a flat brane located at $y = 0$ in a flat bulk

$$ds^2 = -dt^2 + \delta^{ij} dx_i dx_j + dy^2,$$

The equations of motion are

$$\ddot{q} + k^2 q = -\mu^2 q - \beta \phi|_{\text{brane}},$$
$$\ddot{\phi} + k^2 \phi = \phi'' - m^2 \phi,$$

Solve the bulk equation by separating variables

$$\ddot{T}_\rho + \omega^2 T = 0 \quad \text{where} \quad \omega^2 = \rho^2 + m^2 + k^2,$$
$$q(t) = \int_0^\infty d\rho \, C_\rho \, T_\rho(t),$$
$$\phi(t, z) = \int_0^\infty d\rho \, [A_\rho \cos(\rho y) + B_\rho \sin(\rho y)] \, T_\rho(t).$$
The boundary field equation and junction condition relate the coefficients

\[ 2\rho (\rho^2 + m^2 - \mu^2) B_\rho = \beta (\rho^2 + m^2 - \mu^2) C_\rho = \beta^2 A_\rho. \]

**Bound States:**
We impose the condition \( \Im(\rho) > 0 \) and

\[ A_\rho = -iB_\rho. \]

So that \( \phi(y) \to e^{i\rho y} \), which is exponentially decaying.

Combined with the condition above, this yields

\[ 2\rho (\rho^2 + m^2 - \mu^2) + i\beta^2 = 0. \]

This will describe stable oscillations if \( \omega^2 > 0 \), which requires

\[ \beta^2 < 2m\mu^2. \]
• For massive oscillators we get a single bound state provided that the masses and coupling obey $\beta^2 < 2m\mu^2$.

• Compare to non-brane-world case of two coupled oscillators where we can diagonalize the mass matrix.

• The stable bound state allows us to define a vacuum state.
Solutions for an AdS Bulk

[Koyama, Mennim & Wands — hep-th/0504201]

The Randall–Sundrum model: a flat brane \((z = \ell)\) in an AdS bulk

\[
ds^2 = \frac{\ell^2}{z^2} \left( -dt^2 + \delta_{ij} dx_i dx_j + dz^2 \right),
\]

The equations of motion are

\[
\ddot{q} + k^2 q = -\mu^2 q - \beta \phi \big|_{\text{brane}},
\]

\[
\ddot{\phi} + k^2 \phi = \phi'' - \frac{3}{z} \phi' - \frac{m^2 \ell^2}{z^2} \phi,
\]

Separating variables allows us to write the solution as

\[
q(t) = \int_0^\infty d\rho \, C_\rho e^{\pm i \omega t},
\]

\[
\phi(t, z) = \int_0^\infty d\rho \, z^2 [A_\rho J_\nu(\rho z) + B_\rho Y_\nu(\rho z)] e^{\pm i \omega t}.
\]

where \(\omega^2 = \rho^2 + k^2\) and \(\nu = \sqrt{4 + m^2 \ell^2}\).
Bound State Solutions

Consider the simple case where \( m = \mu = 0 \), and use units where \( \ell = 1 \). Bound states satisfy

\[
\rho^3 H_1^{(1)}(\rho) - \frac{\beta^2}{2} H_2^{(1)}(\rho) = 0,
\]

- When \( \beta = 0 \) there is a zero-mode solution \( \rho = 0 \).
- When \( \beta \neq 0 \) this is no longer a solution; for small \( \beta \), it becomes 4 roots, two of which satisfy \( \Im \rho > 0 \):

\[
\rho_1 \approx -\sqrt{\beta} + i \frac{\pi}{16} \beta^{3/2}, \\
\rho_2 \approx i \sqrt{\beta} + \text{no real part},
\]
Recall that \( \omega = \pm \sqrt{\rho^2 + k^2} \).

- The imaginary root \( \rho \approx i\sqrt{\beta} \) is a bound state, but leads to tachyonic instability on long wavelengths.
- The other root \( \rho \approx -\sqrt{\beta} + i\pi\beta^{3/2}/16 \) is a quasi-normal mode describing a metastable state.
- For massive fields we can calculate a critical coupling, as before, below which there is a bound state.
The background line element has the form

$$ds^2 = N(z)^2 \left[ -dt^2 + dz^2 + e^{2Ht}d\vec{x}^2 \right],$$

where $H$ is the Hubble parameter (constant in a de Sitter universe), $\ell$ is the AdS curvature length-scale,

$$N(z) = \frac{H\ell}{\sinh(Hz)},$$

and the brane is located at

$$z_b = H^{-1} \arcsinh(H\ell).$$
Perturbations

Cosmological perturbations are most easily expressed in terms of Mukohyama’s master variable, $\Omega_M$.

$$ds^2 = N(z)^2 \left[ - (1 + 2A) dt^2 + (1 + 2A_{zz}) dz^2 + 2A_z dtdz + a^2 (1 + 2R) d\vec{x}^2 \right].$$

with

$$A = - \frac{1}{6aN^3} \left( 2\Omega''_M - 3 \frac{N'}{N} \Omega_M + \ddot{\Omega}_M - \frac{N^2}{\ell^2} \Omega_M \right),$$

etc.
We perform a rescaling,

\[ \Omega_{\text{M}} = a^3 N^3 \Omega , \]

which obeys a canonical wave equation with a negative effective square-mass.

\[
\frac{\partial^2 \Omega}{\partial t^2} + 3H \frac{\partial \Omega}{\partial t} + k^2 e^{-2Ht} \Omega = \\
\frac{\partial^2 \Omega}{\partial z^2} - 3H \frac{\cosh(Hz)}{\sinh(Hz)} \frac{\partial \Omega}{\partial z} + \frac{4H^2}{\sinh^2(Hz)} \Omega .
\]

We can solve this by separating variables, writing

\[ \Omega(t, z) = \int d\nu \ T_\nu(t) Z_\nu(z) . \]
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\[ \ddot{T} + 3H \dot{T} + \left[ \left( \frac{9}{4} - \nu^2 \right) H^2 + k^2 e^{-2Ht} \right] T = 0, \]

\[ Z'' - 3H \frac{\cosh(Hz)}{\sinh(Hz)} Z' + \left[ \left( \frac{9}{4} - \nu^2 \right) H^2 + \frac{4H^2}{\sinh^2(Hz)} \right] Z = 0, \]

the solutions to which are

\[ T_\nu(t) = e^{-3Ht/2} \left\{ A_\nu \ J_\nu \left( \frac{k}{H} e^{-Ht} \right) + B_\nu \ Y_\nu \left( \frac{k}{H} e^{-Ht} \right) \right\}, \]

\[ Z_\nu(z) = \sinh^2(Hz) \left\{ a_\nu \ P_{\nu - 1/2} \left( \cosh(Hz) \right) \right. \]

\[ + b_\nu \ Q_{\nu - 1/2} \left( \cosh(Hz) \right) \right\}, \]

where \( J \) and \( Y \) are Bessel functions and \( P \) and \( Q \) are associated Legendre functions.
Perturbations on the boundary

The boundary field is the perturbation of the inflaton $\phi$, written as $\delta \phi$. However, it is more convenient to define the, gauge-invariant, Muhkanov-Sasaki variable $Q$, given by

$$Q = \delta \phi - \frac{\dot{\phi}}{H} \psi$$

where $\psi$ is the Newtonian potential. The equation of motion for $Q$ is

$$\ddot{Q} + 3H \dot{Q} + \left[ k^2 e^{-2Ht} + (\eta - 6\varepsilon)H^2 \right] Q =$$

$$\dot{\phi} k^2 \left\{ \ddot{\Omega} + 5H \dot{\Omega} + \left[ k^2 e^{-2Ht} + (6 - 3\varepsilon)H^2 \right] \Omega \right\}$$

where

$$\varepsilon = -\frac{\dot{H}}{H^2} \quad \text{and} \quad \eta = \frac{V''}{H^2}$$
We seek a solution by writing $Q$ in terms of the functions $T_\nu$ as

$$Q = \int d\nu \, C_\nu \, T_\nu(t),$$

Substituting this and the expression for $\Omega$ into the boundary equation of motion gives

$$\int d\nu \left[ \nu^2 - \frac{9}{4} - 6\varepsilon + \eta \right] H^2 C_\nu T_\nu =$$

$$- \frac{\beta k^2}{\kappa \sqrt{6H}} \int d\nu \left\{ 2H \dot{T}_\nu + \left[ \nu^2 - \frac{15}{4} - 3\varepsilon \right] H^2 T_\nu \right\} Z_\nu \bigg|_{\text{brane}}$$

where

$$\beta^2 = \frac{\kappa^2 \phi^2}{6H} \approx \frac{\varepsilon}{3}$$
Large $k$ limit

Changing to conformal time,

$$\int d\nu \left[ \nu^2 - \frac{9}{4} - 6\varepsilon + \eta \right] H^2 C_\nu T_\nu =$$

$$\frac{\beta k^2}{\kappa \sqrt{6H}} \int d\nu \left\{ \frac{2H}{a} \frac{dT_\nu}{d\tau} - \left[ \nu^2 - \frac{15}{4} - 3\varepsilon \right] H^2 T_\nu \right\} Z_\nu \bigg|_{\text{brane}}$$

In the small-wavelength limit we can treat $a$ as approximately constant. Then we can write $T_\nu$ as

$$T_\nu(\tau) = e^{i\omega_\nu \tau}$$

where $\omega_\nu$ must satisfy

$$\omega_\nu^2 - 2Hai\omega_\nu + \left( \nu^2 - \frac{9}{4} \right) H^2 a^2 - k^2 = 0$$
We are interested in the primordial universe where $H\ell \ll 1$, so we use the asymptotic expansion

\[ \frac{Q_{\nu + \frac{1}{2}}}{Q_{\nu - \frac{1}{2}}} \left( \frac{\sqrt{1 + H^2 \ell^2}}{\sqrt{1 + H^2 \ell^2}} \right) \sim \frac{2\nu + 1}{4(\nu + 1)} \frac{1}{H\ell} \]

This gives the condition as

\[ \frac{\beta^2 k^2}{a^2 H^2} \left( \frac{2i\omega}{aH} - \nu^2 \right) \approx - \left( \frac{i\omega}{aH} + 3 \right) \left( \nu^2 - \frac{9}{4} \right) \left( \nu + \frac{1}{2} \right) \]
Defining $\bar{k} = k/aH$ and $\bar{\omega} = \omega/aH$, we have the following coupled system of equations

$$\bar{\omega}^2 - 2i\bar{\omega} + \nu^2 - \frac{9}{4} - \bar{k}^2 = 0,$$

$$2i\beta^2 \bar{k}^2 \bar{\omega} - \beta^2 \bar{k}^2 \nu^2 = -(i\bar{\omega} + 3)\nu^3 - (i\bar{\omega} + 3)\frac{\nu^2}{2}.$$

We can find the roots approximately by expanding in orders of $k$. Four of the roots are

$$\nu = \pm (1 + i)\sqrt{\bar{k}} \pm \frac{i}{\beta^2 \sqrt{\bar{k}}} - \frac{\bar{\omega}^2}{\beta^2 \bar{k}^2},$$

$$\bar{\omega} = \bar{k} + \frac{1 + i}{\beta^2 \sqrt{k}} \quad \text{or} \quad - \bar{k} + \frac{1 - i}{\beta^2 \sqrt{k}},$$
Of these, two are normalizable.

\[ \nu \approx (1 + i)\sqrt{k} + \frac{i}{\beta^2\sqrt{k}} - \frac{\bar{\omega}^2}{\beta^2 \bar{k}^2}, \]

\[ \bar{\omega} \approx \bar{k} + \frac{1 + i}{\beta^2\sqrt{k}} \quad \text{or} \quad - \bar{k} + \frac{1 - i}{\beta^2\sqrt{k}}, \]

Two more roots are:

\[ \nu \approx \mp i\beta^2\bar{k} + \frac{2}{\beta^2} - \frac{i}{2} \]

\[ \bar{\omega} \approx \pm \bar{k}\sqrt{1 + \beta^4} \]

These are both normalisable.

And a final two:

\[ \nu \approx \mp i\bar{k} + \frac{3}{\beta^2} - \frac{i}{2} \]

\[ \bar{\omega} \approx \pm i\beta^2\bar{k} - \frac{\beta^2}{2} \]

These are normalizable, but are rapidly growing and decaying.
We interpret the following modes as being of physical interest:

\[ \nu \approx \pm (1 + i)\sqrt{\bar{k}} \pm \frac{i}{\beta^2 \sqrt{\bar{k}}} , \]
\[ \bar{\omega} \approx \pm \bar{k} \]

corresponding to a free bulk mode uncoupled to a perturbation of the Mukhanov–Sasaki variable \( Q \).

\[ \nu \approx \mp i \beta^2 \bar{k} + \frac{2}{\beta^2} - \frac{i}{2} \]
\[ \bar{\omega} \approx \pm \bar{k} \sqrt{1 + \beta^4} \]

which is a perturbation of the brane and bulk fields. Have a modified dispersion relation, although \( \mathcal{O}(\varepsilon^2) \). Compared with Transplankian modifications these are at a lower energy scale, and do not have to be added by hand.
Conclusions

• Coupled boundary bulk systems have a rich structure of behaviour

• Our toy models show bound states when the fields are massive and the coupling is less than a critical value.

• For scalar fields in a Randall–Sundrum background we can find quasi-normal modes.

• For metric perturbations in brane-world inflation we have determined bound states in the small-wavelength limit.

• There is a possibility of getting modified dispersion relations.

• Future work is to construct the quantum vacuum state for inflation and look for QNMs.